New Bounds and Constraint Programming Models for the Weighted Vertex Coloring Problem

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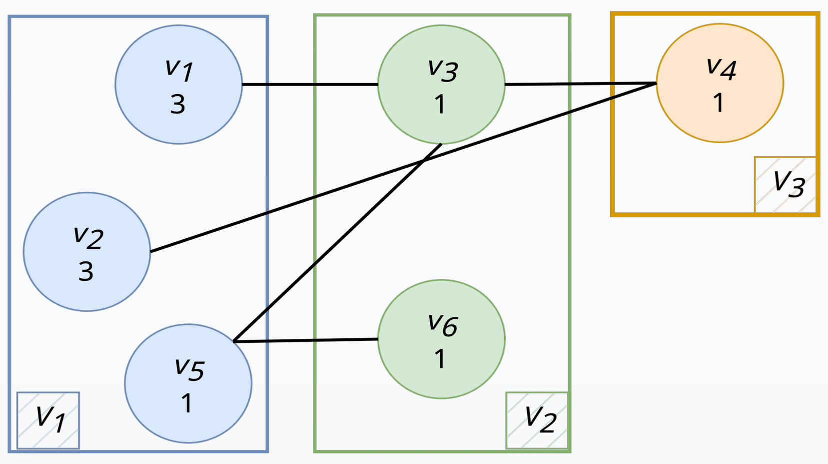
Weighted Vertex Coloring Problem (WVCP)

A WVCP instance is defined by a vertex-weighted graph (G, w) where G is an undirected graph and $w : V \mapsto \mathbb{N}^*$ is the weight function.

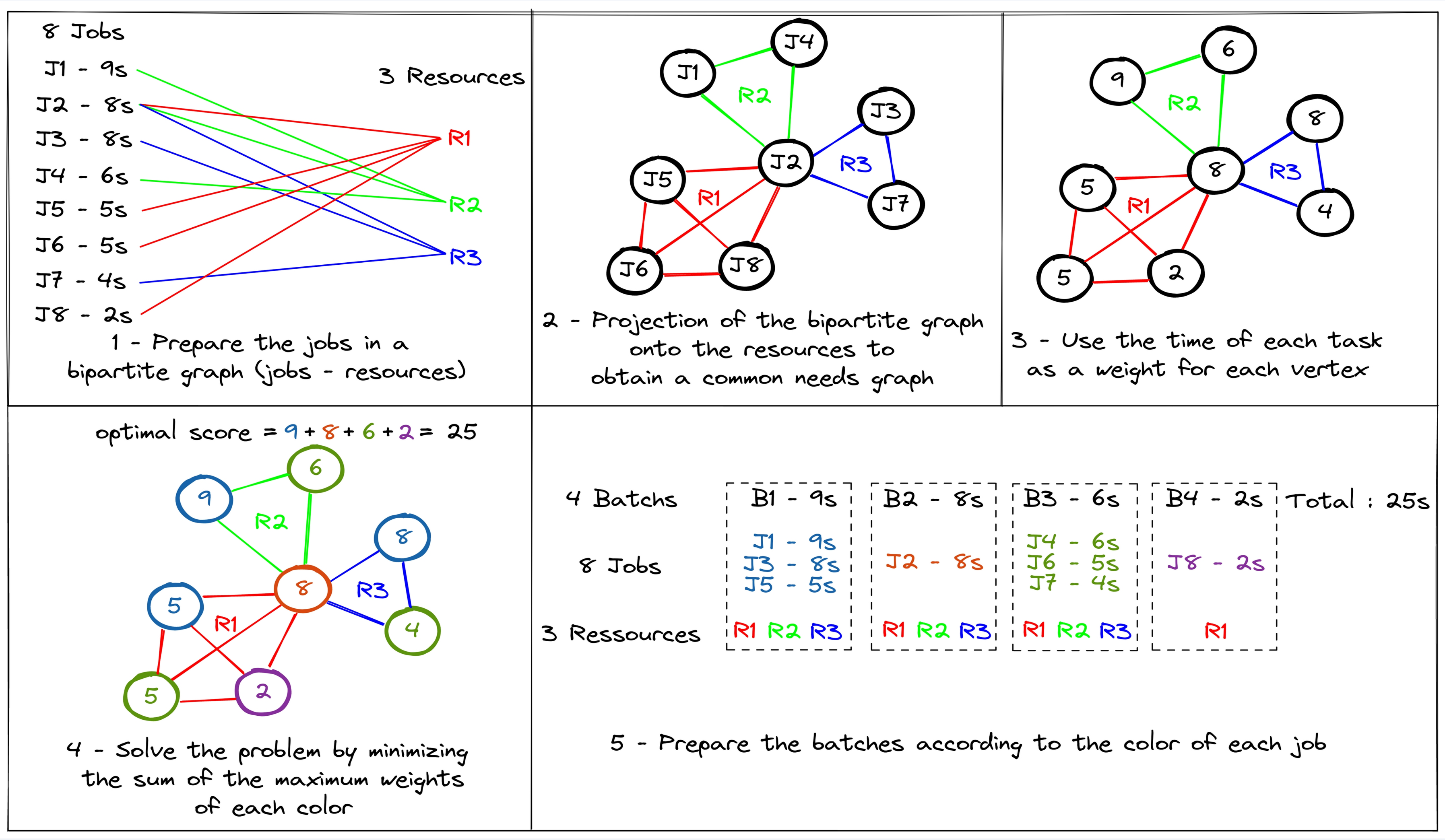
Objective: find a coloring $s = \{V_1, ..., V_k\}$ of G with minimum score $f(s) = \sum_{i=1}^{k} \max_{v \in V_i} w(v)$.

WVCP is NP-hard and has applications in batch scheduling, traffic assignment for satellite communications, matrix decomposition.

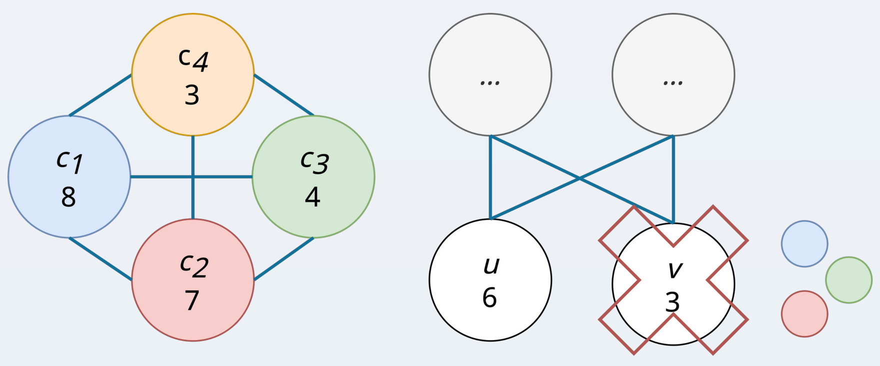
Coloring s. f(s)=3+1+1=5



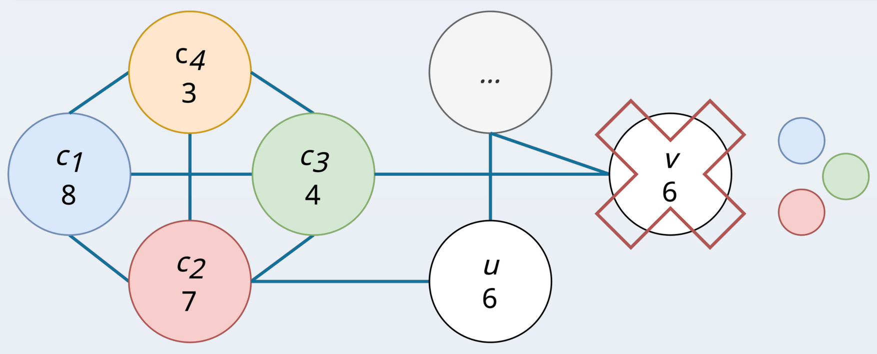
Batch Scheduling as WVCP



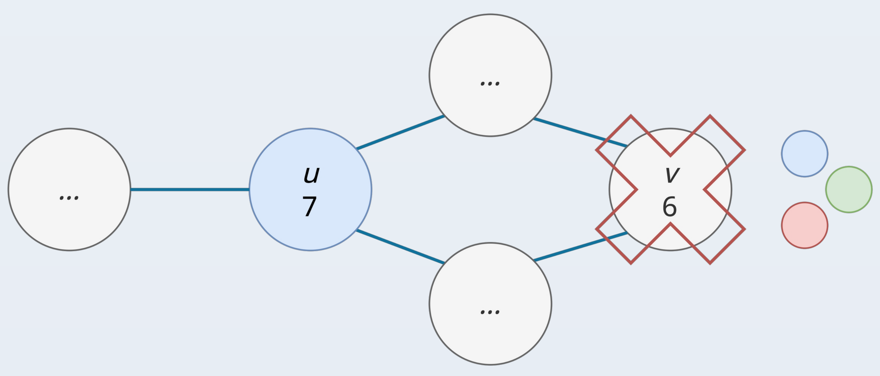
Vertex reduction rules and procedure



Rule R0 of [Wang et al., 2020] considers a clique $C = \{c_1, ..., c_n\}$ s.t. $w(c_i) \geq w(c_{i+1})$. Removes $v \notin C$ if $w(c_{\Delta(v)+1}) > w(v)$.



Rule R1 takes into account that v may have neighbors in C.



Rule R2 adapts [Cheeseman et al., 1991] for GCP. Removes vertex v if its neighborhood is included in the neighborhood of a non-adjacent and no lighter vertex u.

Reduction procedure

1. Extract one clique of maximum weight per vertex using FastWClq [Cai and Lin, 2016].
2. Sort vertices in ascending order of weights.
3. Apply R1 and R2 on each vertex iteratively until fixpoint.

Experiments on 188 benchmark instances

#I number of reduced instances
%V proportion of removed vertices
t(s) average run time in seconds

	#I	%V avg	%V max	t(s)
R0	82	13.4	65	2.6
R1	84	14.7	66.4	3.8
R1+R2	85	15.4	69	4.1
Iterative	85	23.3	80.9	9.8

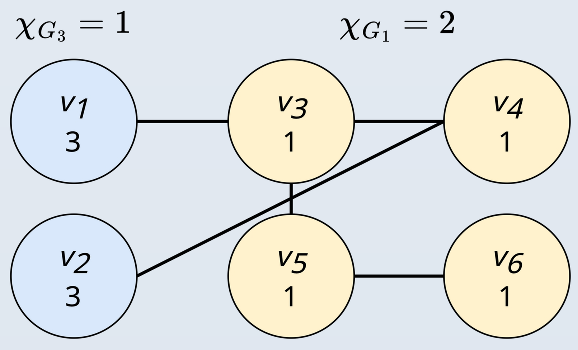
Upper bounds on score and number of colors

An optimal solution to P_k (find a k-coloring to P with minimum score among k-colorings) may not be optimal for P.

$W = \{w(v) \mid v \in V\}$ the set of weights used in $G = (V, E)$.
$G_w = (V_w, E_w)$ the subgraph of G induced by weight $w \in W$.

Theorem. Let s*$= \{V_1, ..., V_k\}$ be an optimal solution to P.

$$k \leq \sum_{w \in W} \chi_{G_w} \quad \text{and} \quad f(s*) \leq \sum_{w \in W} w \times \chi_{G_w}$$



Three CP models for WVCP

Address problem P_k using any valid upper bound k on the number of colors

$V, \Delta(G)+1, \sum_{w \in W}(\chi_{G_w})$, ...

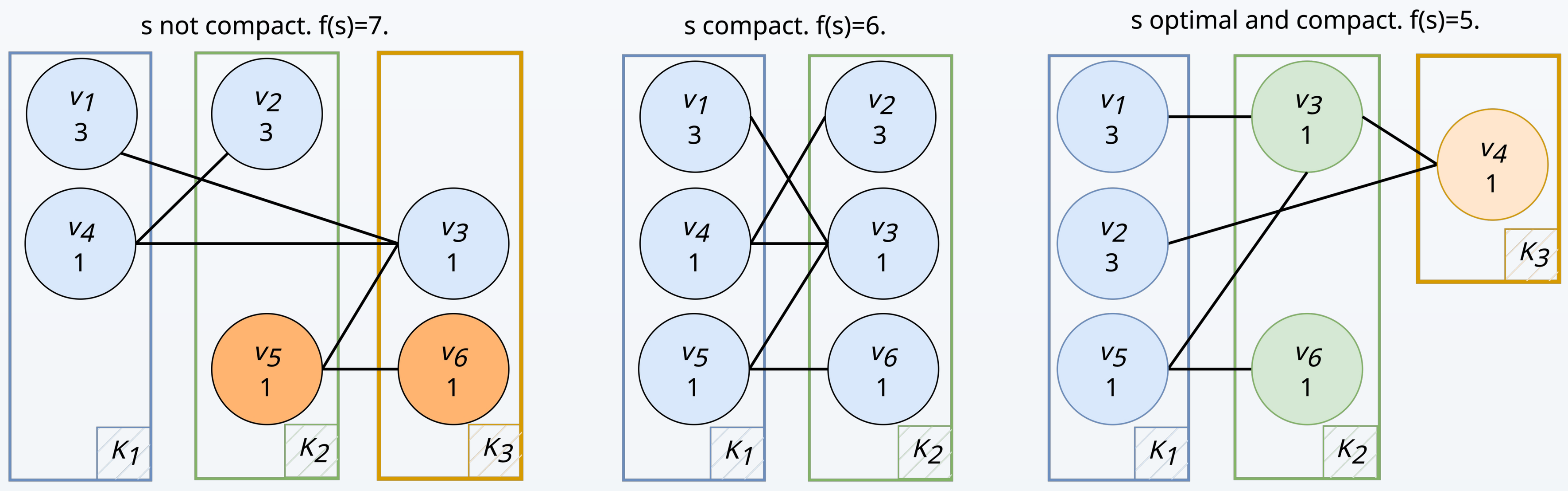
Require a dominance ordering $>_w$ sorting vertices by descending order of weights.

Primal CP model for P_k

Extends classic CP model for GCP.

- Vertices and color dominants as int variables, colors as set variables
- Color dominants sorted using $>_w$ to break symmetries.
- Model solutions (aka. d-solutions) map 1-1 with P_k colorings.

Restricts the search to compact d-solutions wherein no vertex can be moved to a lower-ranked color.



Theorem. Any compact optimal d-solution to $P_{\Delta(G)+1}$ where the domain of each vertex variable v is restricted to $\{1, ..., \Delta(v)+1\}$ is optimal for P.

Global constraint MAX_LEFT_SHIFT

Let $y, x_1, ..., x_n$ be integer variables ($x_i > 0$).
MAX_LEFT_SHIFT$(y, [x_1, ..., x_n])$
holds iff
$$y = \min_{k=1..n+1} (\{k \mid \wedge_{i=1..n} x_i \neq k\}).$$

Ensures solution compactness:
$\forall v_i \in V$: MAX_LEFT_SHIFT$\left(x_i^U, [x_j^U \mid v_j \in N(v_i)]\right)$

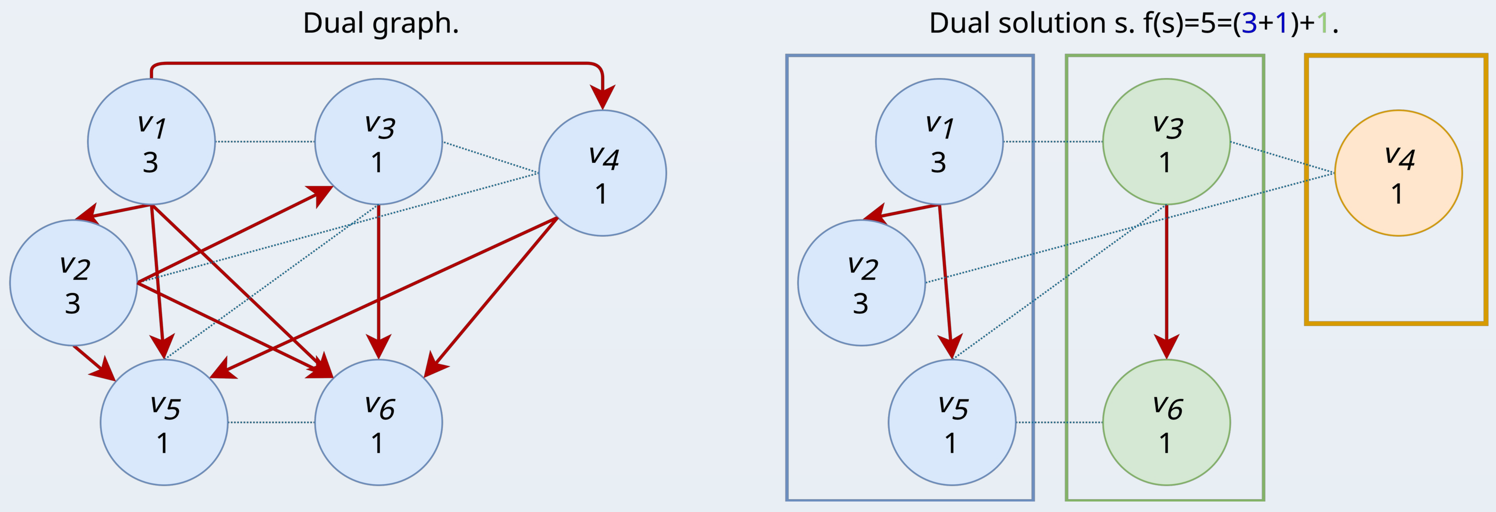
Decomposed using NVALUE [Bessière, 2006].
MAX_LEFT_SHIFT$(y, [x_1, ..., x_n]) \equiv$
$\forall i \in \{1, ..., n\} : y \neq x_i$
$\forall i \in \{1, ..., n\} : z_i \in \{0, ..., n+1\}$
$\forall i \in \{1, ..., n\} : z_i = (y > x_i) \times x_i$
NVALUE$(y, [0, z_1, ..., z_n])$

Dual CP model for P_k

Based on a reduction of WVCP to Maximum Weighted Stable Set Problem [Cornaz et al., 2008].

- Digraph built by complementing E and directing each edge consistently with $>_w$.
- A solution is a forest of simplicial stars spanning disjoint sets of nodes.
- Scored by summing the weights of the target nodes in the stars.

Dual CP model adapts MIP model of [Cornaz et al., 2017].



Joint CP model for P_k

Combines and matches primal and dual solution models.
(J1) Identical colorings and arc inclusion.
(J2) Color dominants and star centers.
(J3) Primal and dual scores.
(J4) Reformulates compactness constraint (P11).

Experiments

Comparison of the CP models.

- Using Minizinc with OR-Tools.
- First-fail on vertex variables with domain bisection.
- 1 hour max/run on a single CPU.
- 1 hour max/run in parallel on 10 threads.

188 instances	primal	primal + P11	dual	joint + J4
nb BKS	101	102/137	79/122	112/132
nb optim	72	76/130	68/111	100/128

10 new optimality proofs including 4 also found with primal without parallelism.

Conclusion

- Effective vertex reduction procedure.
- New upper bounds on score/colors.
- A new global constraint to compact solutions.
- Three competitive and complementary CP models.
- New optimality proofs on benchmarks.

Future work

- Dedicated propagators for MAX_LEFT_SHIFT.
- Hybridization of CP models.